An algorithm generating the (p, q) polygonal tessellation associated to a hyperbolic triangle group

Submission to Instructables *"Made with Math"* contest

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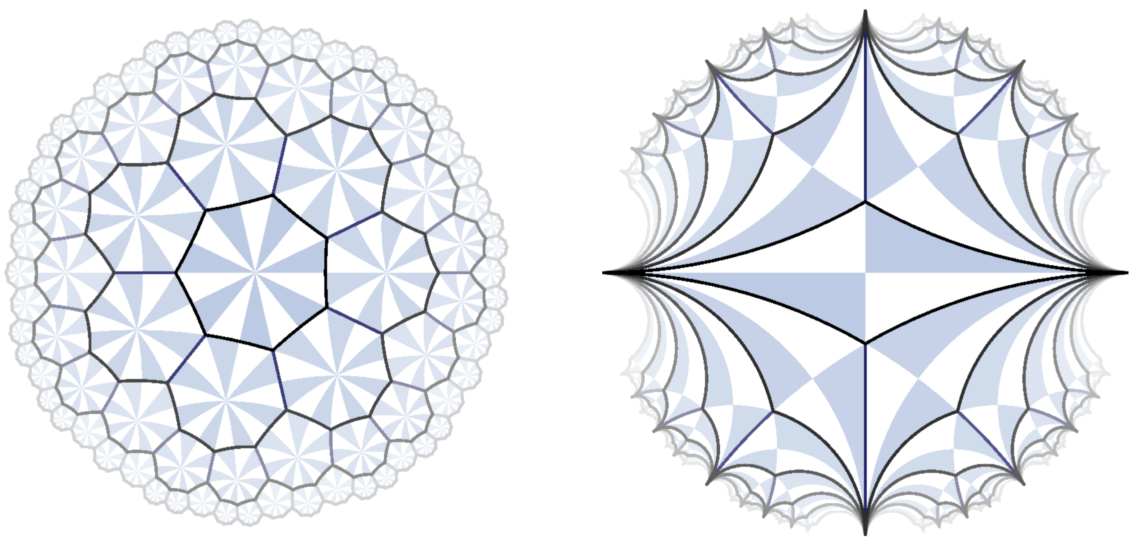

Figure 1: The triangular and polygonal tessellations associated with the groups $D(2, 3, 7)$ (left) and $D(\infty, 3, 2)$ (right). Images of the triangle ABC are filled in blue and images of the triangle $B'AC$ are left white. The polygonal tessellation is outlined in black.

1 The tessellation of the hyperbolic plane associated with the (p, q, r) triangle group

1.1 The (p, q, r) triangular tessellation of the hyperbolic plane

Let p, q, r be three integers $\geqslant 2$ (and possibly infinite). The *von Dyck group* is the group $D = D(p, q, r)$ with generators P, Q, R and relations $P^p = Q^q = R^r = PQR = 1$.

We further assume that $\frac{1}{p} + \frac{1}{q} + \frac{1}{r} < 1$, so that the group D is hyperbolic[5, II.5]. If any of the integers are infinite, we use the convention $1/\infty = 0$, and remove the corresponding relation from the relations defining D. For example, the group $D(\infty, 3, 2)$ is generated by two generators Q, R and the relations $Q^3 = R^2 = 1$; its is isomorphic [6, Chap.VII] to the subgroup of index two in the full modular group $PSL_2(\mathbb{Z})$ generated by $Q = \begin{pmatrix} 0 & -1 \\ 1 & 1 \end{pmatrix}$ and $R = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$.

Tiling and fundamental domain. The group D defines a regular tessellation of the hyperbolic plane $\mathbb{H}$ in the following way. Let ABC be a triangle with vertex angles $\widehat{A} = \pi/p$, $\widehat{B} = \pi/q$ and $\widehat{C} = \pi/r$, and let P, Q, R be the rotations around A, B, C with angle $2\pi/p$, $2\pi/q$, $2\pi/r$. Since P, Q, R satisfy the relations of D, this defines an action of D by rotations of $\mathbb{H}$.

The fundamental domain $D\backslash\mathbb{H}$ is the union of two copies of ABC. For example, let B' be the image of B by the reflexion with axis AC: then the quadrilateral $\mathcal{Q} = ABCB'$ is a fundamental domain of $\mathbb{H}$.

Since $\mathcal{Q}$ is a fundamental domain of $\mathbb{H}/D$, copies of $\mathcal{Q}$ in the tessellation correspond bijectively to elements of D. As an example, since $D(\infty, 3, 2)$ has index two in $PSL_2(\mathbb{Z})$, the triangles (blue and white) in the tiling shown in Figure 1 (right) correspond to the classes of 2×2 matrices with integer coefficients and determinant $+1$.

The standard (p, q, r) triangle. Given three integers $p, q, r \in [2, \infty]$, we compute the vertices of a hyperbolic triangle ABC (in the Poincaré disk model) with angles $\pi/p, \pi/q, \pi/r$.

According to the second law of hyperbolic cosines [9, Theorem 4.2], defining d as the distance in the hyperbolic plane $\mathbb{H}$, one has

$$\cosh d(B, C) = \frac{\cos \frac{\pi}{p} + \cos \frac{\pi}{q} \cos \frac{\pi}{r}}{\sin \frac{\pi}{q} \sin \frac{\pi}{r}}. \tag{1}$$

Write λ for this value (which is known from the integers p, q, r), and likewise μ for the known value for $\cosh d(A, C)$.

If $r < \infty$ we may assume that C is the origin and that $A = \alpha$ lies on the segment $[0, 1]$; then $B = \beta e^{i\pi/r}$ for some real number $\beta \in [0, 1]$. According to the definition of the metric of $\mathbb{H}$, one finds $d(B, C) = \log \frac{1+\beta}{1-\beta}$ and hence $\beta = \sqrt{\frac{\mu-1}{\mu+1}}$. Likewise one has $\alpha = \sqrt{\frac{\lambda-1}{\lambda+1}}$.

Generators of the von Dyck group. Let ABC be a triangle with angles $(\pi/p, \pi/q, \pi/r)$. Then the reflections S_A, S_B, S_C around each edge of ABC generate the triangle group $T^\star(A, B, C)$ [5] with the relations

$$S_A^2 = S_B^2 = S_C^2 = (S_B S_C)^p = (S_C S_A)^q = (S_A S_B)^r = 1. \tag{2}$$

All three reflections can be written as Möbius transforms of $\mathbb{H}$, for example

$$S_A : z \longmapsto \frac{uz + v}{\overline{v}z + \overline{u}}, \quad u = b - a + ab(\overline{a - b});\ v = a\overline{b} - b\overline{a}. \tag{3}$$

The generators of the von Dyck group are the three products $P = S_B S_C$, $Q = S_C S_A$, $R = S_A S_B$. The representations of these as Möbius transforms of $\mathbb{H}$ can be obtained by composing the transforms for S_A, S_B and S_C.

If all three integers p, q, r are finite then P, Q, R are rotations of $\mathbb{H}$ with known centres and angles and can thus be determined explicitly. However writing these as composition of reflections works in all cases, including the case where one of the vertices of ABC is ideal.

1.2 Normal form

The word problem. For two words w, w' on the letters P, P^{-1}, Q, Q^{-1}, R, R^{-1}, we write $w \equiv w'$ if the images of w and w' in D coincide. The *word problem* for D is the following: given a word w, decide if $w \equiv 1$ in D. An algorithm for this problem may also be used to decide if $w \equiv w'$ (by checking whether $w^{-1}w' \equiv 1$).

A possible method for solving the word problem is to give, for each equivalence class of words, a unique normal form, and an algorithm that reduces a word to the equivalent normalized word. For example, the *shortlex* normal form of a word w is the word $w' \equiv w$ which has the smallest number of letters and comes first in the lexicographical ordering, according to some ordering of the letters.

2 Normal form for words in the (p, q) polygonal tessellation

In this section we mark one of the vertices of the triangle ABC, say C, to play a particular role. We are now interested in the images of the point C by the group D, or, equivalently, by the images of the *central polygon*, which is the union of all the triangles of the triangular tessellation that have C as a vertex.

2.1 The polygonal tessellation

We define the *central polygon* as the union $\mathcal{P} = \bigcup R^k(\mathcal{Q})$, where $\mathcal{Q}$ is the central quadrilateral, and the *polygonal tessellation* $\mathcal{T}$ as the tessellation of $\mathbb{H}$ by copies of $\mathcal{P}$. Since the stabilizer of $\mathcal{P}$ in D is the cyclic group $\langle R \rangle$, the cells of $\mathcal{T}$ are in bijection with the right quotient $D/\langle R \rangle$.

Two cases are possible, depending on the values of p and q:

(i) If $p \geqslant 3$ and $q \geqslant 3$ then $\mathcal{P}$ is a $(2r)$-gon, with vertices $A, B, R(A), R(B), \ldots R^{r-1}(A), R^{r-1}(B)$. At each vertex of the form $R^k A$ (resp. $R^k B$), p copies (q copies) of the polygon $\mathcal{P}$ meet.

(ii) If $p = 2$ then $\mathcal{P}$ is a regular r-gon, with vertices $B, R(B), \ldots R^{r-1}(B)$. The points $R^k(A)$ are the midpoints of the edges. The associated tessellation is a tiling of $\mathbb{H}$ by regular r-gons with q polygons meeting at each vertex. (The case $q = 2$ is similar. The case $p = q = 2$ is not hyperbolic.).

The hyperbolic plane $\mathbb{H}$ is tiled by copies of $\mathcal{P}$, with each tile having as its center an image of the point C by the triangle group $D(p, q, r)$. Since $R(\mathcal{P}) = \mathcal{P}$, the tiles also correspond bijectively to elements of D modulo right multiplication by powers of R, that is, to elements of the right quotient set $D/\langle R \rangle$.

2.2 The distance function on tiles.

We define *adjacent* tiles as tiles sharing an edge, and a distance function $d(.,.)$ on tiles such that two adjacent tiles are at a distance 1. Let $\mathcal{B}(n)$ be the ball of radius n around the center tile. For example, according to what precedes, the ball $\mathcal{B}(1)$ contains $2r + 1$ elements if $p, q \geqslant 3$, and $r + 1$ elements if $p = 2$ or $q = 2$.

Let Σ be an alphabet of $2r$ letters written $\left\{R^k P, R^k Q\right\}$ for $k = 0, \ldots, r - 1$. There exists a natural semigroup morpism $\Sigma^\star \to D(p, q, r)$ and, for any word $w \in \Sigma^\star$, the distance $d(w(\mathcal{P}), \mathcal{P})$ is the smallest length of a word $w' \equiv w$.

The naïve tile enumeration algorithm. We can generate a list of all the tiles in the ball $\mathcal{B}(n)$ in the following way:

(i) enumerate the set $\mathcal{W}$ of all words $w \in \Sigma^\star$ of length $\leqslant n$;

(ii) for each such word w, compute $w(C)$, and remove duplicate values.

The reduction with respect to $R^k \equiv 1$ guarantees that the set $\mathcal{W}$ is finite (indeed, it has cardinality $1 + (2r) + \ldots (2r)^n = O(2r)^n$). The second part, however, involves computations in floating-point arithmetic, which are typically inefficient, and may require high precision for larger values of n.

Instead of this algorithm we explain how to compute directly a set of representatives for $\mathcal{B}(n)$, using only exact computations. For this we give an algorithm that reduces words to a normal form. We can then generate only the reduced words by eliminating any word that contains one of the patterns that can be substituted.

2.3 An algorithm generating words in normal form

Proposition 1. *Let $a, b, c = \lfloor p/2 \rfloor, \lfloor q/2 \rfloor, \lfloor (p+q)/2 \rfloor$, and define the following words :*

$$\begin{aligned} W &= P^{a-1} R^{-1} Q^{c-a-1} P^{p-(c-b)-1} R^{-1} Q^{q-b-1} \\ X &= (RQ)^{p-a-1} R (RP)^{q-(c-a)-1} (RQ)^{(c-b)-1} R (RP)^{b-1} \\ Y &= Q^{b-1} P^{c-b-1} R^{-1} Q^{q-(c-a)-1} P^{p-a-1} R^{-1} \\ Z &= (RP)^{q-b-1} (RQ)^{p-(c-b)-1} R (RP)^{(c-a)-1} (RQ)^{a-1} R \end{aligned} \tag{4}$$

Then

(i) the words W, X, Y, Z have length $p + q - 4$;

(ii) the words W, X, Y, Z are equivalent in D to the following expressions:

$$\begin{aligned} W &\equiv (QR)^{p-a} (RP)^{q-(c-a)-1} R (QR)^{(c-b)-1} (RP)^b, \\ X &\equiv RP^a R^{-1} Q^{c-a-1} P^{p-(c-b)-1} R^{-1} Q^{q-b}, \\ Y &\equiv (RP)^{q-b} R (QR)^{p-(c-b)-1} (RP)^{(c-a)-1} R (QR)^a R^{-1}, \\ Z &\equiv Q^b P^{c-b-1} R^{-1} Q^{q-(c-a)-1} P^{p-a}. \end{aligned} \tag{5}$$

(iii) for any integer n, the following relations hold in the group D:

$$W^n Q \equiv Q X^n; \qquad RPW^n \equiv X^n RP; \qquad QY^n \equiv Z^n Q; \qquad RPZ^n \equiv Y^n RP. \tag{6}$$

Proof. For point (i), since we interpret these words as paths in the right quotient $D/\langle R \rangle$, we ignore any tailing letters R or R^{-1}. We can deduce point (ii) from the relations $P^n \equiv (P^{-1})^{p-n} \equiv (QR)^{p-n}$ and likewise $Q^n \equiv (RP)^{q-n}$ and canceling any PQR expressions (and their cyclic rotations) that appear.

The relations in (iii) are conjugacy relations between W and X and between Y and Z. To prove them it is enough to prove the case $n = 1$, which follows from writing all the right-hand sides using (ii). ◁

The cyclic shortlex relation. We consider the $2r$ words R^kP, R^kQ as a directed graph with arrows $P \to Q \to \ldots \to R^{k-1}Q \to P$. For any two such words x, x', we say that x *cyclically precedes* x', and we write $x \prec x'$, if the length of the path $x \to x'$ in this graph is shorter than the length of $x' \to x$. This is not an order relation; in particular, there is ambiguity when x, x' are antipodal vertices. However, since we will generally consider the case where x, x' are at a distance at most 2, this will not be a problem in practice. For example, one can always write $P \prec Q$ and $Q \prec RP$, and for $r \geqslant 3$ these two relations chain without ambiguity.

We extend this ordering to the *cyclic shortlex* ordering on words, defined as follows: $w \prec w'$ if w is strictly shorter than w', or if the words w, w' have the same length and some prefixes vx, vx' where $x \prec x'$ according to the previous definition. For example, for all four relations (6), the left-hand-side precedes the right-hand side for the cyclic shortlex ordering.

Label	Pattern	→ Replacement	Difference
W_1	$RP\,W^n\,Q$	$\to X^n$	2
W_2	$RP\,W^n\,P^a$	$\to X^n\,(RQ)^{p-a-1}R$	$2-(p-2a)$
W_3	$RP\,W^n\,P^{a-1}R^{-1}Q^{c-a}$	$\to X^n\,(RQ)^{p-a-1}R(RP)^{q-(c-a)-1}$	$2-(p+q-2c)$
W_4	$RP\,W^n\,P^{a-1}R^{-1}Q^{c-a-1}P^{p-(c-b)}$	$\to X^n\,(RQ)^{p-a-1}R(RP)^{q-(c-a)-1}(RQ)^{(c-b)-1}R$	$2-(q-2b)$
X_1	$(RQ)^{p-a}$	$\to RP^aR^{-1}$	$p-2a$
X_2	$(RQ)^{p-a-1}R(RP)^{q-(c-a)}$	$\to RP^aR^{-1}Q^{c-a-1}$	$p+q-2c$
X_3	$(RQ)^{p-a-1}R(RP)^{q-(c-a)-1}(RQ)^{c-b}$	$\to RP^aR^{-1}Q^{c-a-1}P^{q-(c-b)-1}$	$q-2b$
X_4	$X^{n+1}\,RP$	$\to RP\,W^{n+1}$	0
Y_1	$Q\,Y^n\,RP$	$\to Z^n$	2
Y_2	$Q\,Y^n\,Q^b$	$\to Z^n\,(RP)^{q-b-1}$	$2-(q-2b)$
Y_3	$Q\,Y^n\,Q^{b-1}P^{c-b}R^{-1}$	$\to Z^n\,(RP)^{q-b-1}(RQ)^{p-(c-b)-1}$	$2-(p+q-2c)$
Y_4	$Q\,Y^n\,Q^{b-1}P^{c-b-1}R^{-1}Q^{q-(c-a)}$	$\to Z^n\,(RP)^{q-b-1}(RQ)^{p-(c-b)-1}R(RP)^{c-a-1}$	$2-(p-2a)$
Z_1	$(RP)^{q-b}$	$\to Q^b$	$q-2b$
Z_2	$(RP)^{q-b-1}(RQ)^{p-(c-b)}$	$\to Q^bP^{c-b-1}R^{-1}$	$p+q-2c$
Z_3	$(RP)^{q-b-1}(RQ)^{p-(c-b)-1}R(RP)^{c-a}$	$\to Q^bP^{c-b-1}R^{-1}Q^{p-(c-a)-1}$	$p-2a$
Z_4	$Z^{n+1}\,Q$	$\to Q\,Y^{n+1}$	0

Table 1: Substitution rules for the quotient $D(p,q,r)/\langle R\rangle$. The "difference" column shows the length difference $|\text{pattern}| - |\text{replacement}|$.

Proposition 2. *The images of the relations in table 1, and their left-translates by powers of R, generate the set of relations in the group D that reduce words for the cyclic shortlex ordering.*

Proof. From the relations (6), we deduce that all the lines in the table are indeed relations in D.

The "difference" column of the table indicates the difference between the length of the pattern and that of its replacement. Using this value, we can check that for all lines of the table, the replacement cyclically precedes the avoided pattern. As an example, if $p = 2a$ is even, then both sides of the X_1 substitution $(RQ)^a \to RP^aR^{-1}$ have the same length a, but $RP \prec RQ$.

We only sketch the proof that these substitutions are enough to generate normal forms in $D/\langle R\rangle$. The infinite words PW^∞, X^∞, QY^∞ and Z^∞, as well as their left-translates by powers of R, are all geodesic paths in $\mathcal{T}$. These paths are also the mediatrices between the points R^kA and R^kB; for example, the relation $X^nRP \equiv RPW^n$, shows that $R^{-1}X^\infty$ is the mediatrix between A and B. Let w be a reducible word with reduction w'. We can proceed by induction to assume that w and w' don't have a common prefix. If w' starts by the letter P and w starts by some letter $L \succ P$, for instance by Q, then the path w crosses the mediatrix $R^{-1}X^\infty$, and therefore it is possible to apply one of the substitutions $R^{-1}X_i$ to w and reduce it for the cyclic shortlex ordering. ◁

Proposition 3. *Let Π be the set of images of the patterns given by table 1 by left-multiplication by powers of R and Π' be the subset of Π formed by removing elements where the "difference" value in table 1 is zero.*

(i) The n-th ring of tiles $\mathcal{B}(n) \smallsetminus \mathcal{B}(n-1)$ corresponds bijectively to the set $\Lambda(n)$ of all words of length n on the $2r$ letters $\left\{R^kP, R^kQ\right\}$ that avoid all patterns in the set Π.

(ii) The outer border of the set $\mathcal{B}(n)$ corresponds bijectively to the set $\Lambda'(n)$ of concatenations wx, where $w \in \Lambda(n)$ and x is one of the letters $\{R^k P, R^k Q\}$, such that wx avoids all patterns in Π'.

Proof. Point (i) is a reformulation of Proposition 2. Point (ii) follows from the observation that, for $w \in \Lambda(n)$ and for any letter x, the edge marked by x of the polygon w is an outer border if, and only if, the polygon wx does not belong to $\mathcal{B}(n)$; this is equivalent to saying that the length of the reduced form of wx is *strictly* larger than n. ◁

2.4 The special cases $p = 2$ or $q = 2$

If $p = 2$ then $Q = P^{-1}R^{-1} = PR^{-1}$ in D; in this case we replace the definition of W by the value

$$W = (R^{-1}P)^{b-1}R^{-1}(R^{-1}P)^{q-b-1}R^{-1} \tag{7}$$

(which is equivalent in the group D to the value (4)). Proposition 1, and in particular the relations (6), still hold.

Replacing $Q = PR^{-1}$ in Table 1, we replace all relations containing non-zero powers of Q or RQ (that is, W_1, W_3, X_1, X_3, all Y_i, Z_2 and Z_4) by the relations W'_i provided in table 2. The relation W'_3 involves the letter Q; this relation is not needed for generating tiles, but only for generating borders according to Proposition 3 (ii). Namely, this relation produces all the segments of the border of $\mathcal{B}(n)$ that border a "blue" triangle according to the coloring used in the figures.

Label	Pattern	→ Replacement	Difference
W'_1	$RP\,W^n\,P$	$\to X^n R$	2
W'_2	$RP\,W^n\,(R^{-1}P)^b$	$\to X^n R(RP)^{q-b-1}R$	$2-(q-2b)$
W'_3	$RP\,W^n\,(R^{-1}P)^{b-1}R^{-1}Q$	$\to X^n R(RP)^{q-b-1}$	$2-(q-2b)$
Y'_1	$Q\,Y^n\,Q$	$\to Z^n$	2
Y'_2	$Q\,Y^n\,(R^{-1}Q)^a$	$\to Z^n R(QR)^{p-a-1}$	$2-(p-2a)$
Y'_3	$Q\,Y^n\,(R^{-1}Q)^{a-1}P$	$\to Z^n R(QR)^{p-a-1}$	$2-(p-2a)$

Table 2: Substitution rules for the quotient $D(p,q,r)/\langle R\rangle$ in the cases $p = 2$ (relations W_i) or $q = 2$ (relations Y_i).

Likewise, in the case where $q = 2$, we have $P = R^{-1}Q$ and define $Y = (R^{-1}Q)^{a-1}R^{-1}(R^{-1}Q)^{p-a-1}R^{-1}$. This produces the relations labeled Y'_i in table 2, where again the relation Y'_3 is used only for borders.

3 Examples

We now present a few explicit examples for small values of p, q, including a method for counting the number of tiles at any given distance from the center tile.

The case $(p = 2, q = 3)$**.** In this case we have $(a,b,c) = (1,1,2)$. In this case, we have $a = 1$ and $c = b+1$. According to the fifth line of Table 1, the pattern (RQ), and hence all words containing a letter R^kQ, are redundant. Therefore all words accepted by the algorithm can be written using only the r letters $G_k = R^kP$ for $k = 0, \dots, r-1$ (we shall sometimes write G_{-k} as a shorthand for G_{r-k}).

We find $W = G_{-2}R^{-1}$ and $X = G_2R$, and hence for any letter L: $W^{n+1}L = G_{-2}G_{-3}^nR^{-1}L$ and $X^{n+1}L = G_2G_3^nRL$.

The expressions from the first column of Table 1, after removing duplicates and multiplying on the left by powers of R, amount to

$$G_kG_0, \quad G_kG_{-1}, \quad G_kG_{-2}G_{-3}^nG_{-2}, \quad G_kG_1, \quad G_kG_2, \tag{8}$$

Thus the polygons of the tiling correspond to words accepted by the regular expression

$$1 + (G_0 + \dots G_{r-1})\big(G_3 + \dots + G_{r-3} + G_{r-2}G_{r-3}^\star(G_3 + \dots + G_{r-4})\big)\big(1 + G_{r-2}G_{r-3}^\star\big). \tag{9}$$

The generating function [3, I.4] for this regular expression is

$$1 + \frac{rz}{1-(r-4)z+z^2} = 1 + r\,z + r(r-4)\,z^2 + r(r-3)(r-5)\,z^3 + O(z^4). \tag{10}$$

This means that there are r tiles at a distance 1 from the center, $r(r-4)$ tiles at a distance 2, and so on.

The smallest value of r such that $1/p+1/q+1/r<1$ is $r=7$. For this value we obtain the OEIS [7] sequence A001354 $(1,7,21,56,147\ldots)$, and for $r=8$ the OEIS sequence A196097 $(1,8,32,120,448\ldots)$. The sequences for larger values of r are not referenced in OEIS.

The case $(p=2, q=4)$**.** We now have $(a,b,c)=(1,2,3)$. All tiles can again be described by words on the r letters $G_k = R^k P$ for $k=0,\ldots,r-1$. From (4) we deduce

$$W = G_{-1}G_{-2}R^{-1}, \quad X \quad = G_2^2, \tag{11}$$

and therefore

$$W^{n+1}L = G_{-1}G_{-2}^{2n+1}R^{-1}L. \tag{12}$$

The patterns avoided are now G_kG_0, G_kG_1, $G_kG_{-1}G_{-2}^{\star}G_{-1}$. Therefore a regular expression matching all words for polygonal tiles is

$$1+(G_0+\ldots+G_{r-1})(G_2+\ldots+G_{r-2}+G_{r-1}G_{r-2}^{\star}(G_2+\ldots+G_{r-3}))^{\star}(1+G_{r-1}G_{r-2}^{\star}) \tag{13}$$

The corresponding generating series is

$$1+\frac{rz}{1-(r-2)z+z^2} \;=\; 1+r\,z+r(r-2)\,z^2+r(r-1)(r-3)\,z^3+O(z^4). \tag{14}$$

For $r=5$ we obtain the OEIS sequence A054888, and for $r=6$ the sequence A001352.

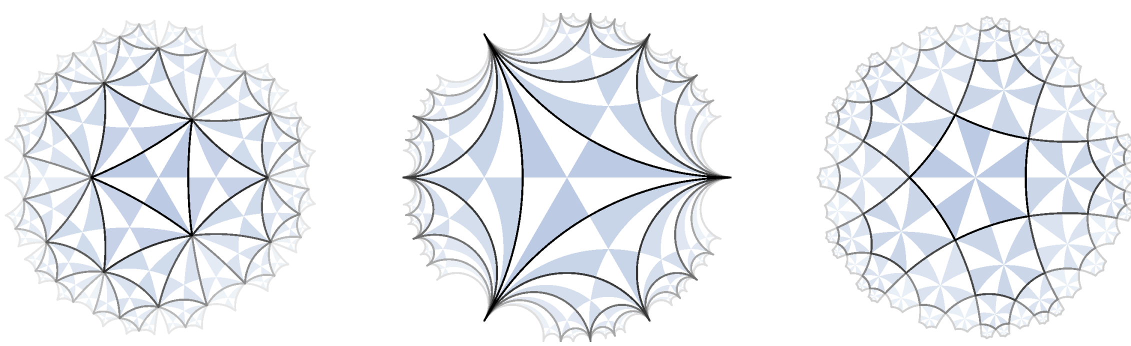

Figure 2: The triangular and polygonal tessellations associated with the groups $D(2,8,3)$ (left), $D(\infty,2,3)$ (center) and $D(2,4,5)$ (right).

The case $(p=2, r\geqslant 5)$**.**

4 Implementation

This algorithm was implemented in the three-dimensional modeling language `OpenSCAD`, with the goal of generating physical models of tilings of the hyperbolic plane in the Poincaré disk model. The same program can also be used to render computer images of these tilings.

The main work is done by a pair of functions:

(i) a function `triangle_group_tessellation_data` generating data (tiles and edges) for a ball of polygons according to the algorithm given in Proposition 3. This function takes as parameters the triple (p,q,r) and the maximum distance n. The special values $p=\infty$ or $q=\infty$ are encoded as zero in the file. The special value $r=\infty$ is currently not supported.

(ii) and a function `klein_quartic_data` generating data in the same format for the tiling of the Klein quartic by 336 hyperbolic triangles of angles $(\pi/2,\pi/3,\pi/7)$. Since this is a finite subset of the $D(2,3,7)$ tiling, this data is actually a constant structure, and therefore the function takes no parameters.

The code also contains modules that convert tessellation data in this format into three-dimensional objects:

(i) `hyperbolic_draw` and `hyperbolic_fill` are low-level functions that draw (as two-dimensional objects) a set of geodesic segments or fill a set of geodesic polygons in $\mathbb{H}$;

(ii) some higher-level functions, such as `tessellation_simple` or `tessellation_cookie`, use these to produce some three-dimensional objects from tessellation data.

Rendering a projection of these objects on the (x, y) plane allows the code to be used to generate computer images of tilings. This is the method that was used to generate all the illustrations in this document.

Given the limitations of the OpenSCAD language, the author needed to rewrite many basic functions, including complex number arithmetic, elementry plane geometry, the Poincaré disk model of the hyperbolic plane, and a crude regular expressions engine. This had an unfortunate effect (despite the author's best efforts) on the size, intelligibility and complexity of the code. None of those basic functions claim to be particularly optimized.

The code was uploaded by the author, as a single file `hyperbolic.scad`, on the Thingiverse website [8] in April 2019. By default, this file produces a cookie cutter that may be used to cut (delicious) cookies and emboss them with the triangles of a hyperbolic (p, q, r) tiling, for user-configured values of (p, q, r). Thanks to OpenSCAD's `use` primitive, the same file may also be used in other designs; for example, the author used it to produce such items as a hyperbolic-themed ventilation grate or a Klein quartic-shaped fruit bowl.

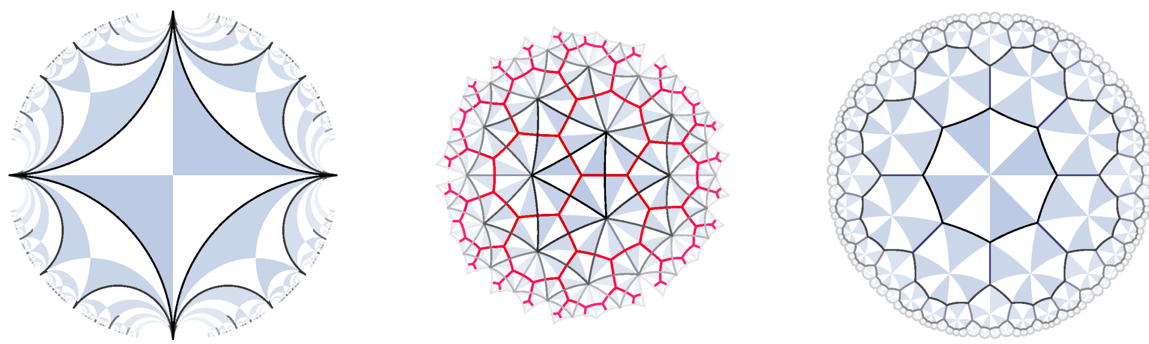

Figure 3: The tessellations associated with the groups $D(\infty, \infty, 2)$, $D(2, 7, 3)$ and $D(3, 3, 4)$. The triangular tiling for $D(2, 7, 3)$ is a translation of the tiling for $D(2, 3, 7)$ (also shown here with red edges).

References

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